

# Hook Length Formulas for Trees by Han's Expansion

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## Abstract

Recently Han obtained a general formula for the weight function corresponding to the expansion of a generating function in terms of hook lengths of binary trees. In this paper, we present formulas for  $k$ -ary trees, plane trees, plane forests, labeled trees and forests. We also find appropriate generating functions which lead to unifications of the hook length formulas due to Du and Liu, Han, Gessel and Seo, and Postnikov.

**Keywords:** hook length formulas for trees,  $k$ -ary trees, plane trees, labeled trees.

## 1 Introduction

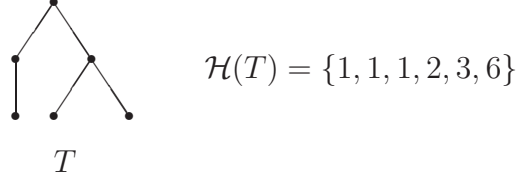
Recently, Han developed an expansion technique for deriving hook length formulas for binary trees. He has shown that given any formal power series  $f(x)$  with  $f(0) = 1$ , one can determine the weight function  $\rho(n)$  that leads to a hook length formula for binary trees. In this paper, we extend Han's technique and obtain the expansion formulas for  $k$ -ary trees, plane trees, plane forests, labeled trees and forests. We find appropriate generating functions that can be used to derive new hook length formulas, some of which can be viewed as unifications of the formulas due to Du and Liu [3], Han [7, 8, 9], Gessel and Seo [5].

Let us give a quick review of the background and terminology. For a tree (or forest)  $T$  the hook length of a vertex  $u$  of  $T$ , denoted by  $h_u$ , is the number of descendants of  $u$  in  $T$  under the assumption that  $u$  is counted as a descendant of itself. The hook length multi-set  $\mathcal{H}(T) = \{h_u : u \in T\}$  of  $T$  is defined to be the multi-set of hook lengths of the vertices  $u$  of  $T$ .

Postnikov [12] discovered the following remarkable hook length formula for binary trees

$$\frac{n!}{2^n} \sum_T \prod_{h \in \mathcal{H}(T)} \left(1 + \frac{1}{h}\right) = (n+1)^{n-1}, \quad (1.1)$$

where the sum ranges over binary trees with  $n$  vertices. Combinatorial proofs of (1.1) have been given by Chen and Yang [2], and Seo [14]. Hook length formulas have been



found for  $k$ -ary trees, plane forests and forests. Du and Liu [3] have obtained the following formulas

$$\sum_T \prod_{h \in \mathcal{H}(T)} \left(a + \frac{1}{h}\right) = \frac{(a+1)}{n!} \prod_{i=1}^{n-1} (kan + a + 1 - i(a - k + 1)) \quad (1.2)$$

and

$$\sum_F \prod_{h \in \mathcal{H}(F)} \left(a + \frac{1}{h}\right) = \frac{(a+1)}{n!} \prod_{i=1}^{n-1} ((2n+1)(a+1) - (a+2)i), \quad (1.3)$$

where  $T$  (resp.,  $F$ ) ranges over  $k$ -ary trees (resp., plane forests) with  $n$  vertices. Liu [10] has given a hook length formula for plane forests with a given degree sequence. Gessel and Seo [5] independently discovered (1.2) and obtained the following formula for forests

$$\sum_F \prod_{h \in \mathcal{H}(F)} \left(1 + \frac{a}{h}\right) = (a+1) \prod_{i=1}^{n-1} ((a+1)n - ai), \quad (1.4)$$

where  $F$  ranges over forests with  $n$  vertices.

Han [9] has found the following formula for binary trees

$$n! \sum_T \prod_{h \in \mathcal{H}(T)} \frac{(z+h)^{h-1}}{2h(2z+h-1)^{h-2}} = z(n+z)^{n-1}, \quad (1.5)$$

where the sum ranges over binary trees with  $n$  vertices. Recall that the polynomials  $z(n+z)^{n-1}$  are the classical Abel polynomials, see Mullin and Rota [11]. The above formula was first proved by Han [9] by induction, and then brought into the framework of his expansion technique [7].

Han's expansion technique for binary trees can be described as follows. Denote by  $\mathbb{K}[[x]]$  the set of formal power series over some field  $\mathbb{K}$ . Define the weight function  $\rho: \mathbb{N}^+ \rightarrow \mathbb{K}$  to be a mapping from the set of positive integers to  $\mathbb{K}$ . Denote by  $B(n)$  the set of binary trees with  $n$  vertices. Han [7] has shown that if the following relation holds

$$1 + \sum_{n \geq 1} \left( \sum_{T \in B(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = f(x), \quad (1.6)$$

then the weight function  $\rho$  is given by

$$\rho(n) = \frac{[x^n]f(x)}{[x^{n-1}]f(x)^2}, \quad (1.7)$$

where  $[x^n]f(x)$  denotes the coefficient of  $x^n$  in the formal power series expansion of  $f(x)$ . The above formula is called the expansion formula for binary trees. Note that each  $T \in B(n)$  ( $n \geq 1$ ) can be decomposed into a triple  $(T', T'', u)$ , where  $T' \in B(m)$  ( $0 \leq m \leq n-1$ ),  $T'' \in B(n-1-m)$  and  $u$  is the root of  $T$  with hook length  $h_u = n$ . Suppose that (1.6) holds. Then we deduce that

$$\begin{aligned} [x^n]f(x) &= \rho(n) \sum_{(T', T'')} \prod_{h \in \mathcal{H}(T')} \rho(h) \prod_{h \in \mathcal{H}(T'')} \rho(h) \\ &= \rho(n) \sum_{m=0}^{n-1} [x^m]f(x) [x^{n-1-m}]f(x), \end{aligned}$$

which implies (1.7).

For example, let  $g(x)$  be defined by the functional equation  $g(x) = \exp\{xg(x)\}$  and  $f(x) = g(2x)$ . Applying the Lagrange inversion formula (see, Stanley [15, Chapter 5]), we get

$$[x^n]xg(x) = \frac{1}{n}[x^{n-1}]e^{nx} = \frac{n^{n-1}}{n!}.$$

Then we have

$$f(x) = \sum_{n \geq 0} (n+1)^{n-1} \frac{(2x)^n}{n!}.$$

By Han's expansion formula for binary trees and the Lagrange inversion formula for the expansion of  $f(x)^2$ , the weight function corresponding to the generating function  $f(x)$  is given by

$$\rho(n) = \frac{[x^n]f(x)}{[x^{n-1}]f(x)^2} = 1 + \frac{1}{n},$$

which leads to Postnikov's formula (1.1).

This paper is organized as follows. In Section 2, we give a straightforward extension of Han's expansion formula to  $k$ -ary trees, and find a generating function whose corresponding hook length formula is a unification of several known formulas. In Section 3, we consider the expansion formulas for plane trees and plane forests. In Section 4, we present expansion formulas and hook length formulas for labeled trees and forests. We conclude this paper with the question of finding combinatorial interpretations of three hook length formulas for plane forests and labeled forests.

## 2 $k$ -ary trees

In this section, we begin with a straightforward extension of Han's expansion formula for binary trees to  $k$ -ary trees. The formula of Yang for  $k$ -ary trees [16] corresponds to the expansion of  $e^x$ . Moreover, we find a generating function, which is defined by a functional equation, which enables us to deduce a hook length formula with one more parameter  $z$  compared with the formula independently due to Du and Liu [3], and Gessel and Seo [5].

Recall that a  $k$ -ary tree is an ordered rooted unlabeled tree where each vertex has exactly  $k$  subtrees in linear order where we allow a subtree being empty. When  $k = 2$ , a  $k$ -ary tree is called a binary tree. Let  $T_k(n)$  denote the set of  $k$ -ary trees with  $n$  vertices.

**Theorem 2.1** *Suppose that we have the following expansion formula for  $k$ -ary trees*

$$1 + \sum_{n \geq 1} \left( \sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = f(x).$$

*Then the weight function  $\rho$  is given by*

$$\rho(n) = \frac{[x^n]f(x)}{[x^{n-1}]f(x)^k}. \quad (2.1)$$

**Example 2.2** *Let  $f(x) = e^x$ . Applying (2.1) we get*

$$\rho(n) = \frac{[x^n]e^x}{[x^{n-1}]e^{kx}} = \frac{1}{nk^{n-1}}.$$

*Hence*

$$\sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{hk^{h-1}} = \frac{1}{n!}. \quad (2.2)$$

The above formula (2.2) was derived by Han [8] for  $k = 2$ . Yang [16] showed that (2.2) holds for general  $k$ . Probabilistic and combinatorial proofs of (2.2) have been given by Sagan [13], and Chen, Gao and Guo [1], respectively.

Below is another hook length formula of Han for binary trees.

**Theorem 2.3** (Han[7], Theorem 6.8) *For  $n \geq 1$ ,*

$$\begin{aligned} & \sum_{T \in B(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + z + (2h - i)a + i)}{2h \prod_{i=1}^{h-2} (2za + 2z + (2h - 2 - i)a + i)} \\ &= \frac{z(a+1)}{n!} \prod_{i=1}^{n-1} (za + z + (2n - i)a + i). \end{aligned} \quad (2.3)$$

The generating function  $f(x)$  for the above expansion is given by the following functional equation

$$g(x) = (a-1)x(1+g(x))^{\frac{2a}{a-1}}$$

and

$$f(x) = (1+g(x))^{z\frac{2a}{a-1}}.$$

To extend Han's formula to  $k$ -ary trees, one needs to find the appropriate extension of the generating function to general  $k$ .

**Theorem 2.4** *Let  $g(x)$  be defined by the functional equation*

$$g(x) = (a-k)x(1+g(x))^{\frac{k(a-1)}{a-k}},$$

*and  $f(x)$  be given by*

$$f(x) = (1+g(x))^{z\frac{a}{a-k}}.$$

*Then the weight function  $\rho$  corresponding to the hook length expansion of  $f(x)$  for  $k$ -ary trees is given by*

$$\rho(n) = \frac{\prod_{i=1}^{n-1} (za + k(a-1)n - i(a-k))}{kn \prod_{i=1}^{n-2} (kza + k(a-1)(n-1) - i(a-k))}, \quad (2.4)$$

*and hence*

$$\begin{aligned} \sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (za + k(a-1)h - i(a-k))}{kh \prod_{i=1}^{h-2} (kza + k(a-1)(h-1) - i(a-k))} \\ = \frac{za}{n!} \prod_{i=1}^{n-1} (za + k(a-1)n - i(a-k)). \end{aligned} \quad (2.5)$$

*Proof.* By the Lagrange inversion formula we obtain

$$\begin{aligned} [x^n]f(x) &= \frac{1}{n} [x^{n-1}] z \frac{a}{a-k} (1+x)^{z\frac{a}{a-k}-1} (a-k)^n (1+x)^{\frac{k(a-1)n}{a-k}} \\ &= \frac{za(a-k)^{n-1}}{n!} \prod_{i=0}^{n-2} \left( z \frac{a}{a-k} - 1 + \frac{k(a-1)n}{a-k} - i \right) \\ &= \frac{za}{n!} \prod_{i=1}^{n-1} (za + k(a-1)n - i(a-k)). \end{aligned}$$

Note that  $[x^n]f(x)^k$  can be easily derived from  $[x^n]f(x)$  by substituting  $z$  with  $kz$ . Consequently,

$$[x^n]f(x)^k = \frac{kza}{n!} \prod_{i=1}^{n-1} (kza + k(a-1)n - i(a-k)).$$

By Theorem 2.1, we obtain the weight function (2.4). This yields (2.5).  $\blacksquare$

The formula (2.5) can be viewed as a unification of several known formulas. Setting  $k = 2$  and substituting  $a$  with  $a + 1$  we obtain Han's formula (2.3). Setting  $z = 1$  in (2.5) yields

$$\sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \left( a - 1 + \frac{1}{h} \right) = \frac{a}{n!} \prod_{i=1}^{n-1} (k(a-1)n + a - i(a-k)),$$

which is equivalent to the formula (1.2) derived independently by Du and Liu [3], and Gessel and Seo [5]. Setting  $a = k$  and  $z = 1$  in (2.5), we obtain

$$\sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \left( k - 1 + \frac{1}{h} \right) = \frac{k^n ((k-1)n + 1)^{n-1}}{n!},$$

which is an extension of Postnikov's hook formula (1.1) to  $k$ -ary trees. Setting  $a = k$  in (2.5), we obtain the following extension of Han's formula (1.5) to  $k$ -ary trees.

**Theorem 2.5** *For  $n \geq 1$ ,*

$$n! \sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \frac{(z + (k-1)h)^{h-1}}{kh(kz + (k-1)(h-1))^{h-2}} = z(z + (k-1)n)^{n-1}. \quad (2.6)$$

Recall that the polynomials  $z(z + (k-1)n)^{n-1}$  are also Abel polynomials (see [11]). Setting  $a \rightarrow \infty$  in (2.5) we deduce the following formula.

**Theorem 2.6** *For  $n \geq 1$ ,*

$$\sum_{T \in T_k(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (kh + z - i)}{kh \prod_{i=1}^{h-2} (kh + k(z-1) - i)} = \frac{z}{n!} \prod_{i=1}^{n-1} (kn + z - i).$$

### 3 Plane trees and plane forests

In this section, we derive hook length expansion formulas for plane trees and plane forests. Then we find certain generating functions to give hook length formulas. Some known formulas can be brought into the framework of the expansion technique.

A plane tree is a rooted unlabeled tree in which the subtrees, assumed to be nonempty, of each vertex are arranged in linear order. A plane forest is a forest of nonempty plane trees which are linearly ordered. Let  $PT(n)$  (resp.,  $PF(n)$ ) denote the set of plane trees (resp., plane forests) with  $n$  vertices.

**Theorem 3.1** *Suppose that the following expansion formula holds for plane trees*

$$\sum_{n \geq 1} \left( \sum_{T \in PT(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = f(x).$$

*Then the weight function  $\rho$  is given by*

$$\rho(n) = \frac{[x^n]f(x)}{[x^{n-1}]\frac{1}{1-f(x)}}, \quad n \geq 1. \quad (3.1)$$

*Proof.* For a plane tree  $T$  with  $n$  ( $n \geq 2$ ) vertices, we obtain a  $j$ -tuple  $(T_1, T_2, \dots, T_j)$  ( $j \geq 1$ ) by deleting the root  $u$  of  $T$ , where  $T_i \in PT(m_i)$  ( $m_i > 0$ ) and  $\sum_{i=1}^j m_i = n - 1$ . Let  $f(n) = [x^n]f(x)$ . Then we have

$$\begin{aligned} f(n) &= \rho(h_u) \sum_{j \geq 1} \sum_{(T_1, T_2, \dots, T_j)} \prod_{i=1}^j \prod_{h \in \mathcal{H}(T_i)} \rho(h) \\ &= \rho(n) \sum_{j \geq 1} \sum_{m_1 + \dots + m_j = n-1} f(m_1) f(m_2) \cdots f(m_j) = \rho(n) [x^{n-1}] \frac{1}{1-f(x)}. \end{aligned}$$

Since  $\rho(1) = f(1)$ , we arrive at (3.1). ■

We proceed to employ (3.1) to derive hook length formulas for plane trees.

**Example 3.2** *Let  $f(x)$  be defined by*

$$f'(x) = \frac{1}{1-f(x)}$$

*with  $f(0) = 0$ . Solving this differential equation we obtain*

$$f(x) = \sum_{n \geq 1} (2n-3)!! \frac{x^n}{n!},$$

*where  $n!! = n(n-2)(n-4) \cdots$  for  $n$  odd and  $(-1)!! = 1$ . By (3.1) we see that  $\rho(n) = \frac{1}{n}$ . Hence we are led to the hook length formula*

$$n! \sum_{T \in PT(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = (2n-3)!!. \quad (3.2)$$

For a plane tree  $T$  with  $n$  vertices, it is well-known that the number of ways to label the vertices of  $T$  with  $\{1, 2, \dots, n\}$ , such that the labeling of each vertex is less than the labelings of its descendants, is equal to  $\frac{n!}{\prod_{h \in \mathcal{H}(T)} h}$  (see, e.g., [5]). Note that  $(2n-3)!!$  equals the number of increasing plane trees on  $n$  vertices, see, e.g., [4].

**Theorem 3.3** For  $n \geq 1$ , we have

$$n! \sum_{T \in PT(n)} \prod_{h \in \mathcal{H}(T)} \left(1 - \frac{1}{h}\right)^{h-1} = (n-1)^{n-1}. \quad (3.3)$$

*Proof.* Let

$$f(x) = \sum_{n \geq 1} (n-1)^{n-1} \frac{x^n}{n!}.$$

It is known that (see, Stanley [15, P. 43])

$$\frac{1}{1-f(x)} = \sum_{n \geq 0} (n+1)^{n-1} \frac{x^n}{n!}.$$

Now, by Theorem 3.1 we find

$$\rho(n) = \frac{(n-1)^{n-1}}{n^{n-1}} = \left(1 - \frac{1}{n}\right)^{n-1},$$

which gives (3.3). ■

We next consider hook length formulas for plane forests.

**Theorem 3.4** Suppose that the following expansion formula holds for plane forests

$$1 + \sum_{n \geq 1} \left( \sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = f(x).$$

Then the weight function  $\rho$  is given by

$$\rho(n) = -\frac{[x^n]f(x)^{-1}}{[x^{n-1}]f(x)}. \quad (3.4)$$

*Proof.* To prove (3.4) we notice that

$$1 + \sum_{n \geq 1} \left( \sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = \left( 1 - \sum_{n \geq 1} \left( \sum_{T \in PT(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n \right)^{-1},$$

that is,

$$\sum_{n \geq 1} \left( \sum_{T \in PT(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n = 1 - \left( 1 + \sum_{n \geq 1} \left( \sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) x^n \right)^{-1}.$$

Utilizing (3.1), it is easy to check (3.4). This completes the proof. ■

Let us consider the above expansion for the exponential function.



**Example 3.5** Let  $f(x) = e^x$ . Then

$$\rho(n) = \frac{(-1)^{n+1} \frac{1}{n!}}{\frac{1}{(n-1)!}} = \frac{(-1)^{n+1}}{n}.$$

Therefore,

$$1 + \sum_{n \geq 1} \sum_{T \in PF(n)} x^n \prod_{h \in \mathcal{H}(T)} \frac{(-1)^{h+1}}{h} = e^x.$$

Equating the coefficients of  $x^n$  yields

$$\sum_{T \in PF(n)} \frac{n!}{\prod_{h \in \mathcal{H}(T)} (-1)^h h} = (-1)^n,$$

which can be restated in terms of plane trees

$$\sum_{T \in PT(n)} \frac{n!}{\prod_{h \in \mathcal{H}(T)} (-1)^h h} = -1. \quad (3.5)$$

Note that the formula (3.5) and a combinatorial proof were given by Yang [16]. The following theorem gives an identity involving Bernoulli numbers.

**Theorem 3.6** Let  $B_n$  be the  $n$ th Bernoulli number. Then we have

$$\sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} B_h = \frac{(-1)^n}{(n+1)!}. \quad (3.6)$$

*Proof.* Let  $f(x) = \frac{e^x - 1}{x}$ . We have

$$\frac{1}{f(x)} = \frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!},$$

By (3.4) we have  $\rho(n) = -B_n$ . Hence

$$1 + \sum_{n \geq 1} \sum_{T \in PF(n)} (-x)^n \prod_{h \in \mathcal{H}(T)} B_h = \frac{e^x - 1}{x}. \quad (3.7)$$

Equating the coefficients of  $x^n$  on both sides of (3.7), we get (3.6). ■

The following theorem for plane forests is a unification of several hook length formulas.

**Theorem 3.7** Let  $g(x)$  be defined by the functional equation

$$g(x) = (a+1)x(1+g(x))^{\frac{2a}{a+1}},$$

and  $f(x)$  be given by

$$f(x) = (1+g(x))^{\frac{za}{a+1}}.$$

Then the weight function  $\rho$  corresponding to the hook length expansion of  $f(x)$  for plane forests is given by

$$\rho(n) = \frac{\prod_{i=1}^{n-1} ((2n-z)a - (a+1)i)}{n \prod_{i=1}^{n-2} ((2n-2+z)a - (a+1)i)}. \quad (3.8)$$

Thus we have

$$\begin{aligned} \sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} ((2h-z)a - (a+1)i)}{h \prod_{i=1}^{h-2} ((2h-2+z)a - (a+1)i)} \\ = \frac{za}{n!} \prod_{i=1}^{n-1} ((2n+z)a - (a+1)i). \end{aligned} \quad (3.9)$$

*Proof.* By the Lagrange inversion formula we get

$$\begin{aligned} [x^n]f(x) &= \frac{1}{n} [x^{n-1}] \frac{za}{a+1} (1+x)^{\frac{za}{a+1}-1} (a+1)^n (1+x)^{\frac{2an}{a+1}} \\ &= \frac{za}{n!} \prod_{i=1}^{n-1} ((2n+z)a - (a+1)i). \end{aligned}$$

Substituting  $z$  with  $-z$  yields

$$[x^n]f(x)^{-1} = \frac{-za}{n!} \prod_{i=1}^{n-1} ((2n-z)a - (a+1)i).$$

By Theorem 3.4, it is easy to verify (3.8) and (3.9). ■

We now consider some special cases of the above formula (3.9). Taking  $z = 1$  in (3.9) we get the formula (1.3) derived by Du and Liu [3]

$$\sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \left( a - 1 + \frac{1}{h} \right) = \frac{a}{n!} \prod_{i=1}^{n-1} ((2n+1)a - (a+1)i).$$

Taking  $a = 1$  in (3.9) we have the following identity.

**Theorem 3.8** For  $n \geq 1$ ,

$$\sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (2h - z - 2i)}{h \prod_{i=2}^{h-1} (2h + z - 2i)} = \frac{z}{n!} \prod_{i=1}^{n-1} (2n + z - 2i). \quad (3.10)$$

Taking  $a = -1$  in (3.9) we deduce the following formula.

**Theorem 3.9** For  $n \geq 1$ ,

$$\sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \frac{(2h - z)^{h-1}}{h(2h - 2 + z)^{h-2}} = \frac{z}{n!} (2n + z)^{n-1}. \quad (3.11)$$

Setting  $z = 2$  in (3.11) we get the a formula equivalent to (3.3).

**Theorem 3.10** For  $n \geq 1$ ,

$$n! \sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \left(1 - \frac{1}{h}\right)^{h-1} = (n+1)^{n-1}. \quad (3.12)$$

Taking  $a \rightarrow \infty$  in (3.9) we are led to the following identity.

**Theorem 3.11** For  $n \geq 1$ ,

$$\sum_{T \in PF(n)} \prod_{h \in \mathcal{H}(T)} \frac{(2h - z - 1)_{h-1}}{h(2h + z - 3)_{h-2}} = \frac{z}{n!} (2n + z - 1)_{n-1},$$

where  $(x)_n = x(x-1) \cdots (x-n+1)$  stands for the falling factorial.

## 4 Labeled trees and forests

In this section, we present the expansion formula for labeled trees and forests. Numerous new hook length formulas are derived. In particular, we derive a unified hook length formula which includes the formula (1.4) derived by Gessel and Seo [5] as its special case. Let  $T(n)$  (resp.,  $F(n)$ ) denote the set of labeled trees (resp., forests) with  $n$  vertices.

**Theorem 4.1** Suppose that the following expansion formula holds for labeled trees

$$\sum_{n \geq 1} \left( \sum_{T \in T(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) \frac{x^n}{n!} = f(x).$$

Then the weight function  $\rho$  is given by

$$\rho(n) = \frac{[x^n]f(x)}{[x^{n-1}]e^{f(x)}}. \quad (4.1)$$

*Proof.* For a labeled tree  $T$  with  $n$  ( $n \geq 2$ ) vertices, let  $u$  be the root of  $T$ . Let  $\{T_1, T_2, \dots, T_j\}$  ( $j \geq 1$ ) be the set of subtrees of  $u$ , and let  $B_i$  be the underlying set of  $T_i$ . Keep in mind that  $\{B_1, \dots, B_j\}$  is a partition of  $[n] - \{u\}$ . Put  $f(n) = [x^n]f(x)$ . Since there are  $n$  choices for  $u$ , we find that

$$f(n) = \rho(n)n \sum_{j \geq 1} \sum_{\{B_1, \dots, B_j\}} f(\#B_1) \cdots f(\#B_j),$$

where  $\{B_1, \dots, B_j\}$  ranges over all set partitions of  $[n-1]$  with  $j$  blocks and  $\#B_i$  denotes the cardinality of  $B_i$ . From the exponential formula (see, e.g., Stanley [15]) it follows that

$$\sum_{j \geq 1} \sum_{\{B_1, \dots, B_j\}} f(\#B_1) \cdots f(\#B_j) = \left[ \frac{x^{n-1}}{(n-1)!} \right] e^{f(x)}.$$

Since  $\rho(1) = f(1)$ , we conclude that (4.1) is true for  $n \geq 1$ . ■

**Theorem 4.2** *Let  $B_n$  be the  $n$ -th Bell number; that is, the number of partitions of  $[n]$ . Then we have*

$$\sum_{T \in T(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{h B_{h-1}} = 1.$$

*Proof.* Let  $f(x) = e^x - 1$ . Then

$$e^{f(x)} = \sum_{n \geq 0} B_n \frac{x^n}{n!}.$$

It follows from (4.1) that

$$\rho(n) = \frac{1}{n B_{n-1}},$$

as desired. ■

**Theorem 4.3** *For  $n \geq 1$ , we have*

$$\sum_{T \in T(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = (n-1)!. \tag{4.2}$$

*Proof.* Let

$$f(x) = \ln \frac{1}{1-x}.$$

Clearly,  $e^{f(x)} = \frac{1}{1-x}$ . By (4.1) we see that  $\rho(n) = \frac{1}{n}$ . This completes the proof. ■

The next theorem gives the hook length expansion formula for forests.

**Theorem 4.4** Suppose that the following expansion formula holds

$$1 + \sum_{n \geq 1} \left( \sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) \frac{x^n}{n!} = f(x).$$

Then the weight function  $\rho$  is given by

$$\rho(n) = \frac{[x^n] \ln f(x)}{[x^{n-1}] f(x)}. \quad (4.3)$$

*Proof.* From the the exponential formula (see, Stanley [15, Chpter 5]), we see that

$$f(x) = 1 + \sum_{n \geq 1} \left( \sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) \frac{x^n}{n!} = \exp \left\{ \sum_{n \geq 1} \left( \sum_{T \in T(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) \frac{x^n}{n!} \right\}.$$

Taking logarithm on both sides yields

$$\ln f(x) = \sum_{n \geq 1} \left( \sum_{T \in T(n)} \prod_{h \in \mathcal{H}(T)} \rho(h) \right) \frac{x^n}{n!}. \quad (4.4)$$

Combining the above relation and the expansion formula (4.1) for labeled rooted trees, we arrive at (4.3). ■

**Theorem 4.5** For  $n \geq 1$ ,

$$\frac{1}{n!} \sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \frac{2(2h-2)!!}{h(2h-3)!!} = \binom{2n}{n}.$$

*Proof.* Let

$$f(x) = \frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

Applying (4.3) we obtain

$$\rho(n) = \frac{2(2n-2)!!}{n(2n-3)!!},$$

as desired. ■

The following formula is a unification of several hook length formulas for forests.

**Theorem 4.6** Let  $g(x)$  be defined by the functional equation

$$g(x) = x(a-1)(1+g(x))^{\frac{a}{a-1}},$$

and  $f(x)$  be given by

$$f(x) = (1 + g(x))^{\frac{za}{a-1}}.$$

Then the weight function  $\rho$  corresponding to the hook length expansion of  $f(x)$  for forests is given by

$$\rho(n) = \frac{\prod_{i=1}^{n-1} (an - (a-1)i)}{n \prod_{i=1}^{n-2} (a(n-1+z) - (a-1)i)}. \quad (4.5)$$

So we have

$$\begin{aligned} \sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \frac{\prod_{i=1}^{h-1} (ah - (a-1)i)}{h \prod_{i=1}^{h-2} (a(h-1+z) - (a-1)i)} \\ = za \prod_{i=1}^{n-1} (a(n+z) - (a-1)i). \end{aligned} \quad (4.6)$$

*Proof.* By the Lagrange inversion formula we get

$$\begin{aligned} [x^n]f(x) &= \frac{1}{n} [x^{n-1}] \frac{za}{a-1} (1+x)^{\frac{za}{a-1}-1} (a-1)^n (1+x)^{\frac{an}{a-1}} \\ &= \frac{za}{n!} \prod_{i=1}^{n-1} (a(n+z) - (a-1)i) \end{aligned}$$

and

$$\begin{aligned} [x^n] \ln f(x) &= \frac{1}{n} [x^{n-1}] \frac{za}{a-1} (1+x)^{-1} (a-1)^n (1+x)^{\frac{an}{a-1}} \\ &= \frac{za}{n!} \prod_{i=1}^{n-1} (an - (a-1)i). \end{aligned}$$

By Theorem 4.4, it is easy to verify (4.5), and so we obtain (4.6). ■

Setting  $z = 1$  in (4.6) we obtain the following formula equivalent to (1.4) derived by Gessel and Seo [5]:

$$\sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \left(1 + \frac{a-1}{h}\right) = a \prod_{i=1}^{n-1} (an - (a-1)i).$$

Setting  $a = 1$  in (4.6) we have the following formula.

**Theorem 4.7** For  $n \geq 1$ ,

$$\sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \left(\frac{h}{h-1+z}\right)^{h-2} = z(n+z)^{n-1}. \quad (4.7)$$

Taking  $a \rightarrow \infty$  in (4.6) leads to the following identity.

**Theorem 4.8** *For  $n \geq 1$ ,*

$$\sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \frac{(h-1)!}{h(h-2+z)_{h-2}} = z(n+z-1)_{n-1}. \quad (4.8)$$

Setting  $z = 1$  in (4.8) yields a formula which is equivalent to (4.2).

**Theorem 4.9** *For  $n \geq 1$ ,*

$$\sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = n!. \quad (4.9)$$

Setting  $z = 2$  in (4.8) we get the following formula.

**Theorem 4.10** *For  $n \geq 1$ ,*

$$\sum_{T \in F(n)} \prod_{h \in \mathcal{H}(T)} \frac{1}{h^2} = \frac{(n+1)!}{2^n}. \quad (4.10)$$

It would be interesting to find combinatorial proofs of the formulas (3.12), (4.9) and (4.10). Notice that the right hand side of (3.12) is equal to the number of forests with  $n$  vertices and for a forest  $F$ ,

$$\frac{n!}{\prod_{h \in \mathcal{H}(F)} h}$$

equals the number of increasing labelings of  $F$ , see Gessel and Seo [5].

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